Erratum: Mixing and moment properties of various GARCH and stochastic volatility models*

Marine Carrasco[†] — University of Rochester
Xiaohong Chen[‡] — New York University

September 28, 2005

There is an error in Definition 3 of Carrasco and Chen (2002). We are very grateful to Mika Meitz and Pentti Saikkonen for pointing out this mistake. However, all the mixing and moment properties of various GARCH, stochastic volatility and ACD models presented in Sections 3, 4, 5 and 6 of Carrasco and Chen (2002) are still correct.

The error is in Subsection 2.3 of Carrasco and Chen (2002); the assumptions given in Definition 3 are not sufficient to establish the conclusion of Proposition 4. The old Definition 3 should be replaced by:

DEFINITION 3. *A process* $\{Y_n, n \geq 0\}$ *with state space* $(\mathcal{Y}, \mathcal{B}(\mathcal{Y}))$ *follows a generalized hidden Markov model with a hidden chain* $\{X_n, n \geq 0\}$ *if*

(i) $\{X_n, n \geq 0\}$ *is a unobserved strictly stationary Markov chain with state space* $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$.

(ii) For all $n \geq 1$, *the conditional distribution of* Y_n *given* $(X_n, Y_{n-1}, X_{n-1}, ..., Y_0, X_0)$ *only depends on* X_n.

(iii) The conditional distribution of Y_n *given* X_n *does not depend on* n.

(iv) $\Pr(X_n \in B_X | X_{n-1}, Y_{n-1}, X_{n-2}, Y_{n-2}, ...) = \Pr(X_n \in B_X | X_{n-1}, Y_{n-1})$, *and* $\Pr(X_n \in B_X | X_{n-1}, Y_{n-1})$ *does not depend on* n.

(v) $\Pr(X_n \in B_X | X_0 = x_0, Y_0 = y_0) = \Pr(X_n \in B_X | X_1 = x_1(x_0, y_0))$, *where* $x_1(x_0, y_0)$ *is a deterministic function of* (x_0, y_0).

---

*We thank the editor and a referee for insightful comments.

[†]University of Rochester, Department of Economics, Harkness 226, Rochester, NY 14620, USA. Email: csco@troi.cc.rochester.edu

[‡]Department of Economics, New York University, 269 Mercer Street, 7th Floor, New York, NY 10003, USA; E-mail: xiaohong.chen@nyu.edu

Note that Conditions (i), (ii) and (iii) are exactly the same as before, and are repeated here for clarity purpose only. Conditions (iv) and (v) are new. In particular, Condition (v) rules out the counter example (1) of Meitz and Saikkonen (2004).

Under the new Definition 3, Proposition 4 in Carrasco and Chen (2002) is still correct. Moreover, the proof of Proposition 4 given in Carrasco and Chen (2002) goes through after the following corrections:

First, the statement that $\{(X_n, Y_n), n \geq 0\}$ is a homogeneous Markov chain needs to be established. It follows from conditions (ii), (iii) and (iv) that

$$\begin{aligned} & \Pr\left(X_n \in B_X, Y_n \in B_Y | X_{n-1}, Y_{n-1}, X_{n-2}, Y_{n-2}, ..., X_0, Y_0\right) \\ = & \int_{B_X} \Pr\left(Y_n \in B_Y | X_n = x, X_{n-1}, Y_{n-1}, ..., X_0, Y_0\right) \Pr\left(X_n = x | X_{n-1}, Y_{n-1}, ..., X_0, Y_0\right) dx \\ = & \int_{B_X} \Pr\left(Y_n \in B_Y | X_n = x\right) \Pr\left(X_n = x | Y_{n-1}, X_{n-1}\right) dx \end{aligned}$$

does not depend on n. This proves that $\{Z_n \equiv (X_n, Y_n), n \geq 0\}$ is a homogeneous Markov chain on $(\mathcal{X} \times \mathcal{Y}, \mathcal{B}(\mathcal{X} \times \mathcal{Y}))$.

Next, for any $z_0 = (x_0, y_0) \in \mathcal{X} \times \mathcal{Y}$, $B = B_X \times B_Y \in \mathcal{B}(\mathcal{X} \times \mathcal{Y})$, we denote $P_Z^n(z_0, B) = \Pr((X_n, Y_n) \in B \mid X_0 = x_0, Y_0 = y_0)$ as the n -step ahead transition probability of the Markov process $\{Z_n, n \geq 0\}$, and $P_X^{n-1}(x_1, B_X) = \Pr(X_n \in B_X | X_1 = x_1)$ as the $(n-1)$ -step ahead transition probability of the Markov process $\{X_n, n \geq 0\}$. By conditions (ii) and (v) we have

$P_Z^n(z_0, B)$
$= \Pr(Y_n \in B_Y \mid X_n \in B_X, X_0 = x_0, Y_0 = y_0) \times \Pr(X_n \in B_X \mid X_0 = x_0, Y_0 = y_0)$
$= \Pr(Y_n \in B_Y \mid X_n \in B_X) \times P_X^{n-1}(x_1, B_X)$

where x_1 is a deterministic function of (x_0, y_0) by (v). Hence, we can write

$$P_Z^n(z_0, dz) = \pi_{Y|X}(y|x)dy \times P_X^{n-1}(x_1, dx),$$

where $\pi_{Y|X}(\cdot|x)$ denotes the conditional distribution of Y_n given $X_n = x$. Using the notations π_Z and π_X introduced in the proof of Proposition 4, we have

$$\begin{aligned} & \|P_Z^n(z_0, \cdot) - \pi_Z(\cdot)\| \\ = & \sup_{|h| \leq 1} \left| \int_{\mathcal{X} \times \mathcal{Y}} h(z) \left[P_Z^n(z_0, dz) - \pi_Z(dz)\right] \right| \\ = & \sup_{|h| \leq 1} \left| \int_{\mathcal{X}} \int_{\mathcal{Y}} h(x, y) \left[\pi_{Y|X}(y|x)dy \times P_X^{n-1}(x_1, dx) - \pi_{Y|X}(y|x)dy \times \pi_X(dx)\right] \right| \end{aligned}$$

$$
\begin{aligned}
&= \sup_{|h|\leq 1} \left| \int_{\mathcal{X}} \left(\int_{\mathcal{Y}} h(x,y)\pi_{Y|X}(y|x)dy \right) \left[P_X^{n-1}(x_1,dx) - \pi_X(dx) \right] \right| \\
&\leq \left\| P_X^{n-1}(x_1,\cdot) - \pi_X(\cdot) \right\|.
\end{aligned}
$$

Hence, the geometric ergodicity of $\{X_n, n \geq 0\}$ implies the geometric ergodicity of $\{Z_n \equiv (X_n, Y_n), n \geq 0\}$ and all the conclusions of Proposition 4 are true.

Note that the conditions imposed by Meitz and Saikkonen (Assumption 1) are slightly different from our new Definition 3. First, their Assumption 1 directly requires that $\{(X_n, Y_n)\}$ is a (homogeneous) Markov chain. Second, their condition (b) of Assumption 1 is slightly more general than our condition (v). Basically instead of imposing our (v), they impose

$$
\Pr(X_n \in \cdot | X_0, Y_0) = \Pr(X_n \in \cdot | X_{n-j}) \quad \text{for some} \quad j \geq 0 \text{ for all } n > j,
$$

where X_{n-j} is a deterministic function of (X_0, Y_0). However, this generalization is not needed for the applications in Carrasco and Chen (2002).

In Carrasco and Chen (2002), the conclusions of Proposition 4 are used only to establish the geometric ergodicity of the augmented GARCH(1,1) model described by Equations (2) and (3) that we reproduce here:

$$
\begin{aligned}
\varepsilon_t &= \sqrt{h_t}\eta_t \quad , \ t = 0, 1, ... \qquad (2) \\
\Lambda(h_t) &= c(e_t)\Lambda(h_{t-1}) + g(e_t) \qquad (3)
\end{aligned}
$$

where $\Lambda(\cdot)$ is increasing and continuous with domain $[0, +\infty)$, $\{\eta_t\}$ satisfies Condition η (including i.i.d. (0,1), independent of h_0), and e_t is some measurable function of η_{t-1} and satisfies Condition e (including strictly stationary).

Here we check that the model described by (2) and (3) satisfies Conditions (i) - (v) of the new Definition 3. Let $X_t = h_t$ and $Y_t = \varepsilon_t$.

(i) As Λ is invertible, it follows from (3) that $\{h_t\}$ can be considered as a Markov chain on its own.

(ii) The conditional distribution of ε_t given $(h_t, \varepsilon_{t-1}, h_{t-1}, ...)$ depends only on h_t because $\{\eta_t\}$ are independent.

(iii) The conditional distribution of ε_t given h_t does not depend on t by the stationarity of η_t.

(iv) If $(h_{t-1}, \varepsilon_{t-1})$ is known, then $\eta_{t-1} = \varepsilon_{t-1}/\sqrt{h_{t-1}}$ is known and so is e_t. Hence conditional on $(h_{t-1}, \varepsilon_{t-1})$, h_t is completely known. It follows that

$$\begin{aligned} & \Pr\left(X_t \in B_X | Y_{t-1}, X_{t-1}, Y_{t-2}, X_{t-2}, ...\right) \\ = \;& \Pr\left(h_t \in B_X | \varepsilon_{t-1}, h_{t-1}, \varepsilon_{t-2}, h_{t-2}, ...\right) = \Pr\left(h_t \in B_X | h_{t-1}, \varepsilon_{t-1}\right) \end{aligned}$$

is either 0 or 1.

(v) For the reasons outlined in (iv), we have

$$\Pr\left(X_t \in B_X | Y_0, X_0\right) = \Pr\left(h_t \in B_X | \varepsilon_0, h_0\right) = \Pr\left(h_t \in B_X | h_1\left(\varepsilon_0, h_0\right)\right)$$

where h_1 is some deterministic function of (ε_0, h_0).

Hence, the model (2)-(3) satisfies all the conditions of new Definition 3.

It should be emphasized that, because the extra Conditions (iv) and (v) are automatically satisfied by the various augmented GARCH(1,1) models (as shown above), all the results in Section 3 of Carrasco and Chen (2002) are still correct. Moreover, all the results in Sections 4, 5, and 6 of Carrasco and Chen (2002) on higher-order models including the GARCH(p, q), stochastic volatility and ACD models are also correct since they are established without applying Proposition 4.

References

Carrasco, M. and X. Chen (2002) "Mixing and moment properties of various GARCH and stochastic volatility models", *Econometric Theory*, 18, 17-39.

Meitz, M. and P. Saikkonen (2004) "Ergodicity, mixing, and existence of moments of a class of Markov models with applications to GARCH and ACD models", unpublished manuscript, University of Helsinki.